



# Effects of spin viscosity on ferrofluid duct flow profiles in alternating and rotating magnetic fields

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## Abstract

Closed form analytical equations are obtained for ferrofluid flow, spin velocity and volumetric flow rate in alternating and rotating uniform magnetic fields including the effects of spin viscosity.

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*Keywords:* Ferrofluids; Magnetic fluid; Spin viscosity; Effective viscosity

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## 1. Ferrofluids in AC magnetic fields

Recent analysis [1,2] and measurements [3] have shown the anomalous behavior of ferrofluids in AC magnetic fields, whereby in linearly polarized or rotating magnetic fields the effective fluid viscosity can be increased or decreased and the ferrofluid can be pumped but the flow direction can reverse as a function of magnetic field amplitude, frequency, and direction. This anomalous behavior can be explained using the governing fluid mechanical linear and angular momentum conservation equations including a non-symmetric viscous stress tensor. Most of the past work has taken the spin viscosity to be zero. This work extends this past work by examining the effects of non-zero spin viscosity.

We examine simple cases where the applied magnetic fields along and transverse to a duct axis are spatially uniform and vary sinusoidally with time with either linear or elliptical polarizations. In the uniform magnetic field, the magnetization characteristic depends on fluid spin velocity but does not depend on fluid translational velocity. The magnetization force density along the duct axis is then zero while the magnetic torque density is non-zero as magnetization and magnetic fields are not collinear due to magnetic relaxation as well as due to spatially varying fluid spin velocity.

## 2. Governing magnetic equations

The magnetic relaxation equation for a ferrofluid undergoing simultaneous magnetization, convection with mass-average velocity  $\mathbf{v}$ , and reorientation due to particle spin  $\omega$  is [4]

$$\frac{\partial \mathbf{M}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{M} - \omega \times \mathbf{M} + \frac{1}{\tau} [\mathbf{M} - \chi_0 \mathbf{H}] = \mathbf{0}, \quad (1)$$

where  $\tau$  is the relaxation time constant and  $\chi_0$  is the effective magnetic susceptibility which, for simplicity, is assumed constant. We apply Eq. (1) to a planar ferrofluid layer confined between rigid walls of gap width  $d$ . The imposed axial magnetic field  $H_z$  and transverse magnetic flux density  $B_x$  are spatially uniform and vary sinusoidally with time at radian frequency  $\Omega$ . Because the fields are uniform with  $y$  and  $z$  coordinates, field components can only vary with the

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$x$ -coordinate. Gauss’s law for the magnetic flux density and Ampere’s law for the magnetic field intensity with zero current density then require the imposed fields to be spatially uniform independent of  $x$ . However, the resulting magnetization causes  $x$ -dependent  $B_z$  and  $H_x$  components. The magnetic field and flux density are thus of the form

$$\mathbf{B} = \text{Re}\{(\hat{B}_x \mathbf{i}_x + \hat{B}_z(x) \mathbf{i}_z) e^{i\Omega t}\}, \quad \mathbf{H} = \text{Re}\{(\hat{H}_x(x) \mathbf{i}_x + \hat{H}_z \mathbf{i}_z) e^{i\Omega t}\}, \quad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{2}$$

The corresponding magnetic force and torque densities for an incompressible ferrofluid are

$$\mathbf{f} = \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H}, \quad \mathbf{T} = \mu_0(\mathbf{M} \times \mathbf{H}) = \mu_0(-M_x H_z + M_z H_x) \mathbf{i}_y. \tag{3}$$

Along the duct ( $z$ ) axis,  $H_z$  is uniform so that  $f_z = 0$ . In the small spin velocity limit where  $\omega_y \tau \ll 1$ , the approximate time-average magnetic torque density is [1,2]

$$\langle T_y \rangle \approx T_0 + \alpha \omega_y \tag{4}$$

$$T_0 = -\chi_0 \frac{\text{Re}\{[\chi_0(\Omega\tau)^2 + j\Omega\tau((\Omega\tau)^2 + 1 + \chi_0)][\hat{H}_z \hat{B}_x^*]\}}{[1 + \chi_0 + (\Omega\tau)^2]^2 + (\chi_0\Omega\tau)^2},$$

$$\alpha = \frac{\chi_0\tau}{2} \frac{(|\hat{B}_x|^2/\mu_0)[(\Omega\tau)^2 - 1] + \mu_0|\hat{H}_z|^2[(\Omega\tau)^2 - (1 + \chi_0)^2]}{[1 + \chi_0 + (\Omega\tau)^2]^2 + (\chi_0\Omega\tau)^2}. \tag{5}$$

### 3. Governing fluid mechanical equations

In the creeping flow limit the time-averaged linear and internal angular momentum equations for the translational velocity component  $v_z$  and spin velocity component  $\omega_y$  reduce to

$$(n + \zeta) \frac{d^2 v_z}{dx^2} + 2\zeta \frac{d\omega_y}{dx} - \frac{\partial p'}{\partial z} = 0, \quad \eta' \frac{d^2 \omega_y}{dx^2} - 2\zeta \left( \frac{dv_z}{dx} + 2\omega_y \right) + \langle T_y \rangle = 0, \tag{6}$$

where  $\zeta$  is the vortex viscosity,  $\eta$  is the dynamic viscosity,  $p' = p + \rho g x + \mu_0 |\hat{M}_x|^2 / 4$  is the dynamic pressure incorporating gravity and magnetization, and  $\eta'$  is the spin viscosity. The solutions to Eq. (6) using Eqs. (4) and (5) with zero translational and spin velocity boundary conditions at the  $x = 0$  and  $x = d$  boundaries,  $v_z[x = 0] = v_z[x = d] = 0$ ,  $\omega_y[x = 0] = \omega_y[x = d] = 0$ , are:

$$v_z(x) = \frac{x(x-d)}{2\eta_{\text{eff}}} \frac{\partial p'}{\partial z} + \frac{(\eta_{\text{eff}} - \eta - \zeta)}{2\kappa(\eta + \zeta)\eta_{\text{eff}}} \left\{ \frac{\partial p'}{\partial z} \left[ \frac{(1 + \cos \kappa d)(1 - \cos \kappa x) - \sin \kappa d \sin \kappa x}{\sin \kappa d} d \right] \right.$$

$$\left. + \frac{\eta_{\text{eff}} T_0}{d\zeta} \left[ \frac{(1 - \cos \kappa d)(2x - d + d \cos \kappa x) - d \sin \kappa d \sin \kappa x \sin \kappa x}{\sin \kappa d + 2 \frac{(\eta_{\text{eff}} - \eta - \zeta)}{\kappa d (\eta + \zeta)} (1 - \cos \kappa d)} \right] \right\}, \tag{7}$$

$$w_y(x) = \frac{\eta_{\text{eff}} - \eta - \zeta}{4\zeta} \left\{ \frac{1}{\eta_{\text{eff}}} \frac{\partial p'}{\partial z} \left[ \frac{\sin \kappa d (2x - d(1 - \cos \kappa x)) - d(1 + \cos \kappa d) \sin \kappa x}{\sin \kappa d} \right] \right.$$

$$\left. + \frac{T_0}{\zeta} \left[ \frac{(1 - \cos \kappa d) \sin \kappa x - \sin \kappa d (1 - \cos \kappa x)}{\sin \kappa d + 2 \frac{(\eta_{\text{eff}} - \eta - \zeta)}{\kappa d (\eta + \zeta)} (1 - \cos \kappa d)} \right] \right\}, \tag{8}$$

where

$$\kappa = \left( \frac{\alpha \frac{4\zeta\eta}{\eta + \zeta}}{\eta'} \right)^{1/2} = \left( \frac{4\zeta^2 \eta_{\text{eff}}}{(\eta_{\text{eff}} - \eta - \zeta)(\eta + \zeta)\eta'} \right)^{1/2}, \quad \eta_{\text{eff}} = \eta + \frac{\alpha\zeta}{\alpha - 4\zeta} \tag{9}$$

The effective viscosity  $\eta_{\text{eff}}$  is that defined from the translational velocity solution of Eq. (7) when the spin viscosity  $\eta'$  is neglected in Eq. (6) corresponding to the first term on the right-hand side of Eq. (7) [1,2]. Note that the parameter  $\kappa$

is imaginary when  $0 < \eta_{\text{eff}} < \eta + \zeta$  indicating exponential functions in the flow and spin velocity profiles. Otherwise,  $\kappa$  is pure real and the flow and spin velocity profiles have trigonometric variations which allow flow reversals over the duct thickness  $d$ . The resultant volumetric flow rate per unit depth of channel is independent of  $T_0$ :

$$Q = \int_0^d v_z(x) dx = -\frac{d^3}{12\eta_{\text{eff}}} \frac{\partial p'}{\partial z} \left\{ 1 + 6 \frac{\eta_{\text{eff}} - \eta - \zeta}{(\kappa d)^2 (\eta + \zeta)} \left[ 2 - \frac{\kappa d (1 + \cos \kappa d)}{\sin \kappa d} \right] \right\} \quad (10)$$

The flow rate can be positive or negative as a function of  $\kappa d$  corresponding to forward or backward flows. Note that with careful evaluation of Eqs. (7), (8) and (10), singular solutions only result for  $\kappa d = 2n\pi$ ,  $n = 1, 2, \dots$ . Careful evaluation for  $\eta_{\text{eff}} = 0$  results in finite solutions while  $\eta_{\text{eff}} = \zeta + \eta$  is not physical as it is equivalent to  $\alpha = \infty$  which requires infinite magnetic fields.

Figs. 1–4 illustrate representative flow profiles for various values of  $\kappa d$ ,  $\eta'$ ,  $T_0$ ,  $\eta_{\text{eff}}$  and  $\zeta = 3\phi\eta/2$  where  $\phi$  is the magnetic particle volume fraction taken to be 10% in the plots. The important non-dimensional spin viscosity parameter in these plots is  $\eta'/(\eta d^2)$  which from dimensional analysis is of order  $(d_p/d)^2$  where  $d_p$  is the ferrofluid particle diameter, typically of order 10 nm [4]. For macroscopic ducts of order 1 mm,  $\eta'/(\eta d^2) \propto (d_p/d)^2 = 10^{-10}$ . The resulting spin velocity profile then generally has a very thin boundary layer near the duct walls at  $x = 0$  and  $x = d$  with very little effect on the bulk profile. However, such a thin boundary layer can still have a large effect on the shear stress at the walls. To emphasize the effects of spin viscosity in the plots, we consider a microfluidic duct with width  $d$  of order  $1 \mu\text{m}$  so that  $\eta'/(\eta d^2) \propto (d_p/d)^2 = 10^{-4}$ . For such small duct widths spin viscosity affects the entire flow region.

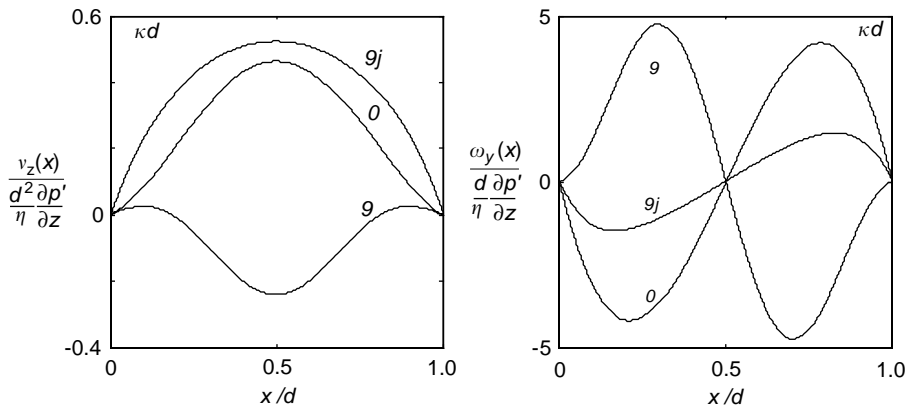


Fig. 1. Effect of  $\kappa d$  on translational (left) and spin (right) velocity profiles for  $T_0 = 0$ . These plots are for  $\zeta/\eta = 0.15$  and  $\eta'/\eta d^2 = 5 \times 10^{-4}$ . The values of  $\kappa d$  correspond to  $\eta_{\text{eff}}/\eta$  of  $-1.15$  ( $\kappa d \approx 9$ ),  $0$  ( $\kappa d = 0$ ), and  $0.38$  ( $\kappa d \approx 9j$ ).

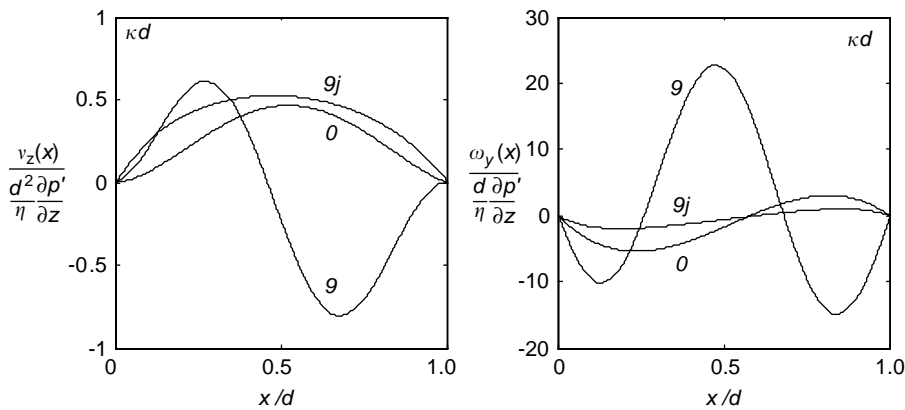


Fig. 2. Effect of  $\kappa d$  on translational (left) and spin (right) velocity profiles for  $T_0 = 0.1d \partial p' / \partial z$ . These plots are for  $\zeta/\eta = 0.15$  and  $\eta'/\eta d^2 = 5 \times 10^{-4}$ . The values of  $\kappa d$  correspond to  $\eta_{\text{eff}}/\eta$  of  $-1.15$  ( $\kappa d \approx 9$ ),  $0$  ( $\kappa d = 0$ ), and  $0.38$  ( $\kappa d \approx 9j$ ).

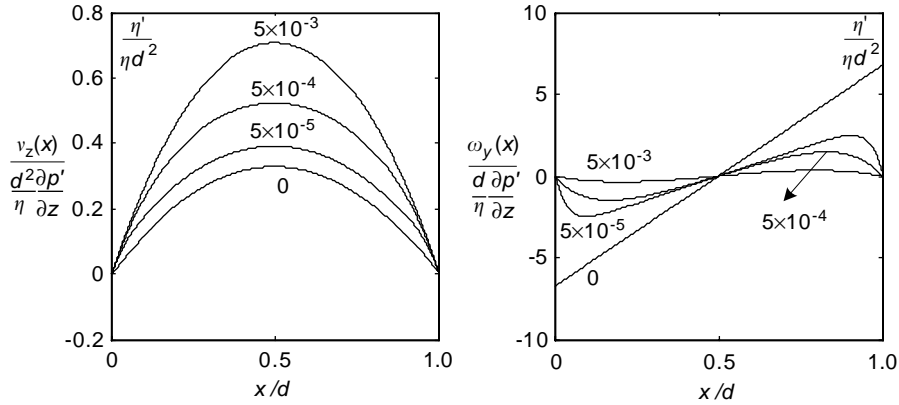


Fig. 3. Effect of spin viscosity on translational (left) and spin (right) velocity profiles for  $T_0 = 0$ . These plots are for  $\zeta/\eta = 0.15$  and  $\eta_{\text{eff}}/\eta = 0.38$  ( $\kappa d$  is imaginary). Notice that as the spin viscosity tends to zero the zero spin-viscosity solution is approached.

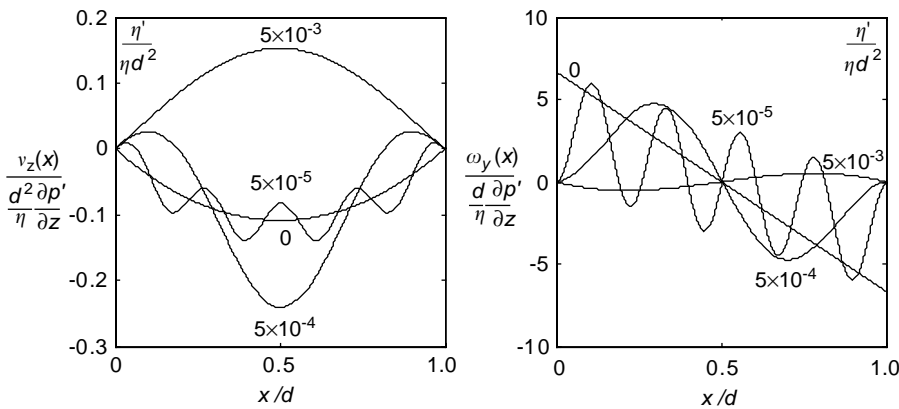


Fig. 4. Effect of spin viscosity on translational (left) and spin (right) velocity profiles for  $T_0 = 0$ . These plots are for  $\zeta/\eta = 0.15$  and  $\eta_{\text{eff}}/\eta = -1.15$  ( $\kappa d$  is real). Notice that as the spin viscosity tends to zero the solutions are oscillatory and approach the zero spin-viscosity solution.

**Acknowledgements**

This work has been supported by the US National Science Foundation Grant CTS-0084070. Carlos Rinaldi has been supported by a National Science Foundation Graduate Research Fellowship.

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